Solution 1:

a) Let T is injective, so for all $v \in Ker(T)$, then $T(v) = 0_W$ 0,5

$\longrightarrow T(v) = T(0_V)$ 0,5 (Because T is injective)

$\longrightarrow v = 0_V$ 0,5

Hence $Ker(T) = \{0_V\}$ 0,25

b) Conversely:

we assume that $Ker(T) = \{0_V\}$ 0,5

and let $v_1, v_2 \in V$ such that 0,25

$T(v_1) = T(v_2)$

$\longrightarrow T(v_1 - v_2) = T(v_1) - T(v_2) = 0_W$ 0,5

$\longrightarrow v_1 - v_2 \in Ker(T) = \{0_V\}$ 0,5

$\longrightarrow v_1 - v_2 = 0_V \longrightarrow v_1 = v_2$ 0,5

Hence T is injective 0,25

Solution 2:

1) We calculate $A^3 - A^2 + A - I$:

$$A^2 = A \times A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$ 0,5

$$A^3 = A^2 \times A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$ 0,5

So:

$A^3 - A^2 + A - I =$ 1

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0_{M_3(\mathbb{R})}$$

2) Let us express A^{-1} as a function of A^2, A and I

we have:

$A^3 - A^2 + A - I = 0$ 0,5

$\longrightarrow A(A^2 - A + I) = I$ 0,5

and

$(A^2 - A + I)A = I$ 0,5

So: $A^{-1} = A^2 - A + I$. 0,5

3) Let us express A^4 as function of A^2, A and I:

we have:

$A^3 - A^2 + A - I = 0$ 0,5

$\longrightarrow A^3 = A^2 - A + I$ 0,5

$\longrightarrow A.A^3 = A(A^2 - A + I)$ 0,25

$\longrightarrow A^4 = A^3 - A^2 + A$

$= (A^2 - A + I) - A^2 + A$ 0,5

$= I$ 0,25

Solution 03:

1) Let us show that f is linear:

Let $\alpha, \beta \in \mathbb{R}$ and $u(x_1; y_1; z_2)$; $v = (x_2; y_2; z_2) \in \mathbb{R}^3$. We show that:

$$f(\alpha u + \beta v) \stackrel{?}{=} \alpha f(u) + \beta f(v) \quad 0{,}5$$

We have:

$$f(\alpha u + \beta v) \underset{0{,}25}{=} f\big(\alpha(x_1, y_1, z_2) + \beta(x_2; y_2, z_2)\big) \underset{0{,}25}{=} f(\underbrace{\alpha x_1 + \beta x_2}_{X}; \underbrace{\alpha y_1 + \beta y_2}_{Y}; \underbrace{\alpha z_1 + \beta z_2}_{Z})$$

$$= (-2X + Y + Z;\ X - 2Y + Z;\ X + Y - 2Z) \quad 0{,}25$$

$$= \big(\alpha(-2x_1 + y_1 + z_1);\ \alpha(x_1 - 2y_1 + z_1);\ \alpha(x_1 + y_1 - 2z_1)\big) + \quad 0{,}25$$

$$\big(\beta(-2x_2 + y_2 + z_2);\ \beta(x_2 - 2y_2 + z_2);\ \beta(x_2 + y_2 - 2z_2)\big)$$

$$= \alpha f(x_1, y_1, z_1) + \beta f(x_2; y_2; z_2) \quad 0{,}25 \quad = \alpha f(u) + \beta f(v). \quad 0{,}25$$

2) Ker f:

we have:

$$\mathrm{Ker}(f) = \{(x,y,z) \in \mathbb{R}^3 \,/\, f(x,y,z) = 0_{\mathbb{R}^3}\} \quad 0{,}25$$

$$= \left\{(x,y,z) \in \mathbb{R}^3 \,/\, \begin{cases} -2x + y + z = 0 \\ x - 2y + z = 0 \\ x + y - 2z = 0 \end{cases}\right\} \quad 0{,}25$$

$$= \left\{(x,y,z) \in \mathbb{R}^3 \,/\, \begin{cases} -2x + y + z = 0 \\ -3y + 3z = 0 & (2L_2 + L_1) \\ 3y - 3z = 0 & (2L_3 + L_1) \end{cases}\right\} \quad 0{,}25$$

$$= \left\{(x,y,z) \in \mathbb{R}^3 \,/\, \begin{cases} -2x + y + z = 0 \\ y = z \end{cases}\right\} \quad 0{,}25 \quad = \{(x,y,z) \in \mathbb{R}^3 \,/\, y = z = x\} \quad 0{,}25$$

$$= \{(x,x,x) \,/\, x \in \mathbb{R}\} \quad 0{,}25 \quad = \{x(1,1,1) \,/\, x \in \mathbb{R}\} \neq \{0_{\mathbb{R}^3}\}$$

$\Longrightarrow f$ is not injective 0,25. Hence f is not bijective 0,25

So $\{v = (1,1,1)\}$ is a spanning set of Ker f. (0,25)

Then: $\dim \operatorname{Ker} f = 1$ (0,2)

* By dimension theorem: (0,15)

$$\dim \mathbb{R}^3 = \dim \operatorname{Ker}(f) + \dim(\operatorname{Im} f)$$

$$\rightarrow \dim \operatorname{Im} f = \dim \mathbb{R}^3 - \dim \operatorname{Ker}(f)$$

$$\rightarrow \dim \operatorname{Im} f = 3 - 1 = 2$$ (0,25)

3) Giving a basis of Im f:

We have:

$$\operatorname{Im} f = \{(-2x+y+z ; x-2y+z ; x+y-2z) \mid x,y,z \in \mathbb{R}\}$$

$$= \{x(-2,1,1) + y(1,-2,1) + z(1,1,-2) \mid x,y,z \in \mathbb{R}\}$$

So $\{v_1 = (-2,1,1) ; v_2 = (1,-2,1) ; v_3 = (1,1,-2)\}$ is a spanning set of Im f. But $\dim \operatorname{Im} f = 2$ (0,1) just then we take two vectors

Let $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha v_1 + \beta v_2 = 0_{\mathbb{R}^3}$$

we have:

$$\alpha \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} -2\alpha + \beta = 0 \\ \alpha - 2\beta = 0 \\ \alpha + \beta = 0 \end{cases} \xrightarrow{(3)} \begin{cases} \beta = -\alpha \\ -3\alpha = 0 \end{cases}$$

$$\rightarrow \alpha = \beta = 0$$

So $\{v_1, v_2\}$ is (0,1) linearly independent set. Hence $\{v_1, v_2\}$ is a basis of Im f. (0,5)

4) $D = \operatorname{Mat}_B(f)$:

we have:

$f(e_1) = f(1,0,0) = (-2,1,1)$ and $f(e_2) = f(0,1,0) = (1,-2,1)$ and $f(e_3) = f(0,0,1) = (1,1,-2)$ (0,25 ×3)

So

$$D = \operatorname{Mat}_B(f) = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$ (0,25)

* $M = \operatorname{Mat}_B(f^2) = A \circ A = A^2$ (0,25)

$$= \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{pmatrix}$$ (0,25)

5) $f^2(x,y,z)$:

Let $(x,y,z) \in \mathbb{R}^3$, we have:

$$f^2(x,y,z) = A^2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$ (0,5)

$$(6x-3y-3z ; -3x+6y-3z ; -3x-3y+6z)$$ (0,5)